

# Lecture 27

## Array Antennas, Fresnel Zone, Rayleigh Distance

Our world is beset with the dizzying impact of wireless communication. It has greatly impacted our lives.<sup>1</sup> Wireless communication is impossible without using antennas. Hence it is important to design these communication systems with the proper antennas so that they operate with utmost efficiency and sensitivity. We have seen that a simple Hertzian dipole has low directivity in Section 25.3. The radiation pattern looks like that of a donut, and the directivity of the antenna is 1.5. Hence, for point-to-point communications, much power is wasted. However, the directivity of antennas can be improved if a group or array of dipoles can work cooperatively together by using constructive and destructive interferences. They can be made to constructively interfere in the desired direction, and destructively interfere in other directions to enhance the directivity of the array of dipoles. Since the far-field approximation of the radiation field can be made, and the relationship between the far field and the source is a Fourier transform relationship, clever engineering can be done borrowing knowledge from the signal processing area. After understanding the far-field physics, one can also understand many optical phenomena, such as how a laser pointer works. Many textbooks have been written about array antennas some of which are [157, 158].

### 27.1 Linear Array of Dipole Antennas

Antenna array can be designed so that the constructive and destructive interference in the far field can be used to steer the direction of radiation of the antenna, or the far-field radiation pattern of an antenna array. This is because the far field of a source is related to the source by a Fourier transform relationship as shown in the previous lecture (see Subsection 26.1.1). The relative phases of the array elements can be changed slowly in time with respect to the operating frequency so that the beam of an array antenna can be steered in real time.

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<sup>1</sup>I cannot imagine a day that I do not receive messages from my relatives and friends in Malaysia and Singapore. It has made the world look smaller, and more transparent.

This has important applications in, for example, air-traffic control. It is to be noted that if the current sources are impressed current sources, they can be regarded as the input to Maxwell's equations. Then the fields are the output of the system, and we are dealing with a linear time-invariant system here whereby linear system theory can be used. For instance, we can use Fourier transform to analyze the problem in the frequency domain. The time domain response then can be obtained by inverse Fourier transform. This is provided that the current sources are impressed and they are not affected by the fields that they radiate.

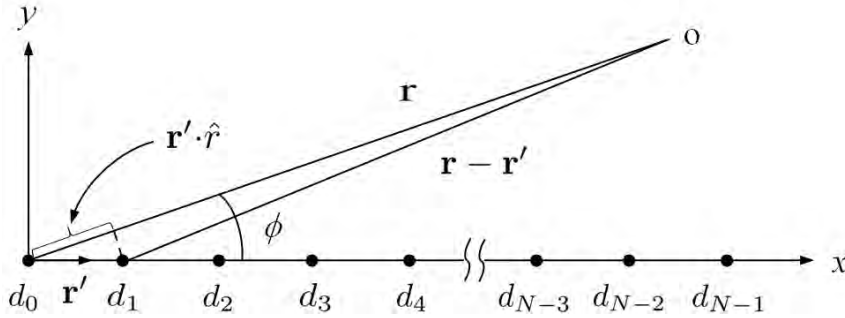


Figure 27.1: Schematic of a dipole array where the dipoles are aligned on the  $x$  axis, and we observe the field on the  $xy$  plane. To simplify the math, the far-field approximation can be used to find its far field or radiation field.

To gain physical insight into how constructive and destructive interference works for an antenna array, we assume a simple linear dipole array as shown in Figure 27.1. First, without loss of generality and for simplicity to elucidate the physics, we assume that this is a linear array of point Hertzian dipoles aligned on the  $x$  axis. The current can then be described mathematically as follows:

$$\begin{aligned} \mathbf{J}(\mathbf{r}') = \hat{z}I[A_0\delta(x') + A_1\delta(x' - d_1) + A_2\delta(x' - d_2) + \dots \\ + A_{N-1}\delta(x' - d_{N-1})]\delta(y')\delta(z') \end{aligned} \quad (27.1.1)$$

Again for simplicity, all the dipoles are pointing in the  $z$  axis. The far field can be found using the approximate formula derived in the previous lecture, viz., (26.1.3) reproduced below:

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu e^{-j\beta r}}{4\pi r} \iiint_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') e^{j\beta \cdot \mathbf{r}'} \quad (27.1.2)$$

To reiterate, the above implies that the far field is related to the Fourier transform of the current source  $\mathbf{J}(\mathbf{r}')$ .

### 27.1.1 Far-Field Approximation of a Linear Array

The vector potential on the  $xy$ -plane in the far field, using the sifting property of delta function, yields the following equation for  $\mathbf{A}(\mathbf{r})$  using (27.1.2),

$$\mathbf{A}(\mathbf{r}) \cong \hat{z} \frac{\mu I l}{4\pi r} e^{-j\beta r} \iiint d\mathbf{r}' [A_0 \delta(x') + A_1 \delta(x' - d_1) + \dots] \delta(y') \delta(z') e^{j\beta \mathbf{r}' \cdot \hat{\mathbf{r}}} \quad (27.1.3)$$

In the above, for simplicity, we will assume that the observation point is on the  $xy$  plane, or that  $\mathbf{r} = \boldsymbol{\rho} = \hat{x}x + \hat{y}y$  where  $\boldsymbol{\rho}$  is the position vector in the  $xy$  plane. Thus,  $\hat{\mathbf{r}} = \hat{x} \cos \phi + \hat{y} \sin \phi$ . Also, since the sources are aligned on the  $x$  axis, then  $\mathbf{r}' = \hat{x}x'$ , and  $\mathbf{r}' \cdot \hat{\mathbf{r}} = x' \cos \phi$ . Consequently,  $e^{j\beta \mathbf{r}' \cdot \hat{\mathbf{r}}} = e^{j\beta x' \cos \phi}$ . By so doing, the far field of a linear array is

$$\mathbf{A}(\mathbf{r}) \cong \hat{z} \frac{\mu I l}{4\pi r} e^{-j\beta r} [A_0 + A_1 e^{j\beta d_1 \cos \phi} + A_2 e^{j\beta d_2 \cos \phi} + \dots + A_{N-1} e^{j\beta d_{N-1} \cos \phi}] \quad (27.1.4)$$

Next, for further simplification, we let  $d_n = nd$ , implying an equally spaced array with distance  $d$  between adjacent elements. Then we let  $A_n = e^{jn\psi}$ , assuming a progressively increasing phase shift between different elements. Such an antenna array is called a *linear phase array*. Thus, (27.1.4) in the above becomes

$$\mathbf{A}(\mathbf{r}) \cong \hat{z} \frac{\mu I l}{4\pi r} e^{-j\beta r} [1 + e^{j(\beta d \cos \phi + \psi)} + e^{j2(\beta d \cos \phi + \psi)} + \dots + e^{j(N-1)(\beta d \cos \phi + \psi)}] \quad (27.1.5)$$

With the simplifying assumptions, the above series can be summed in closed form because it is a series of the form  $1 + x + x^2 + x^3 + \dots + x^{N-1}$ .

### 27.1.2 Radiation Pattern of an Array

The above(27.1.5) can be summed in closed form using the formula

$$\sum_{n=0}^{N-1} x^n = \frac{1 - x^N}{1 - x} \quad (27.1.6)$$

Then in the far field,

$$\mathbf{A}(\mathbf{r}) \cong \hat{z} \frac{\mu I l}{4\pi r} e^{-j\beta r} \frac{1 - e^{jN(\beta d \cos \phi + \psi)}}{1 - e^{j(\beta d \cos \phi + \psi)}} \quad (27.1.7)$$

Ordinarily, as shown previously in (26.1.11),  $\mathbf{E} \approx -j\omega(\hat{\theta}A_\theta + \hat{\phi}A_\phi)$ . But since  $\mathbf{A}$  is  $\hat{z}$  directed,  $A_\phi = 0$ . Furthermore, on the  $xy$  plane,  $E_\theta \approx -j\omega A_\theta = j\omega A_z$ . As a consequence,

$$\begin{aligned} |E_\theta| &\doteq |E_0| \left| \frac{1 - e^{jN(\beta d \cos \phi + \psi)}}{1 - e^{j(\beta d \cos \phi + \psi)}} \right|, \quad \mathbf{r} \rightarrow \infty \\ &= |E_0| \left| \frac{\sin\left(\frac{N}{2}(\beta d \cos \phi + \psi)\right)}{\sin\left(\frac{1}{2}(\beta d \cos \phi + \psi)\right)} \right|, \end{aligned} \quad (27.1.8)$$

The factor multiplying  $|E_0|$  above is also called the *array factor*. The above can be used to plot the far-field pattern of an antenna array.

Equation (27.1.8) has an array factor that is of the form

$$\frac{|\sin(Nx)|}{|\sin x|}$$

This function appears in digital signal processing frequently, and is known as the digital sinc function [159]. The reason why this is so is because the far field is proportional to the Fourier transform of the current. The current in this case a finite array of Hertzian dipole, which is a product of a box function and infinite array of Hertzian dipole. The Fourier transform of such a current, as is well known in digital signal processing, is the digital sinc.<sup>2</sup>

Plots of  $|\sin(3x)|$  and  $|\sin x|$  are shown as an example and the resulting  $\frac{|\sin(3x)|}{|\sin x|}$  is also shown in Figure 27.2. The function peaks (also called the principal maximum) when both the numerator and the denominator of the digital sinc vanish. The value can be found by Taylor series expanding both the numerator and the denominator at the point where they both vanish. This happens when  $x = n\pi$  for integer  $n$ . Such an apparent singularity is called a removable singularity.

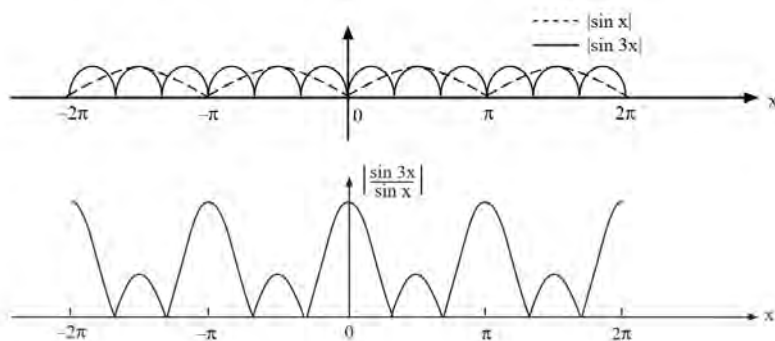


Figure 27.2: Plot of the digital sinc,  $\frac{|\sin 3x|}{|\sin x|}$ . It peaks at points where both the numerator and denominator vanish.

In equation (27.1.8),  $x = \frac{1}{2}(\beta d \cos \phi + \psi)$ . We notice that the *maximum* of the array radiation pattern in (27.1.8) would occur if  $x = n\pi$ , or if

$$\beta d \cos \phi + \psi = 2n\pi, \quad n = 0, \pm 1, \pm 2, \pm 3, \dots \quad (27.1.9)$$

<sup>2</sup>All good electrical engineers should know that the Fourier transform of an infinite impulse train in the time domain is an impulse train in the frequency domain. Hence, the Fourier transform of an infinitely long array of point sources is an infinitely long array of delta functions in the spectral Fourier space. The Fourier transform of a box function is a sinc function. Hence, the Fourier transform of a finite size array of dipoles is the convolution of the sinc function with an infinitely long array of delta function, yielding the digital sinc.

The *zeros* or *nulls* of the radiation pattern will occur at  $Nx = n\pi$ , or

$$\beta d \cos \phi + \psi = \frac{2n\pi}{N}, \quad n = \pm 1, \pm 2, \pm 3, \dots, \quad n \neq mN \quad (27.1.10)$$

For example,

**Case I.**  $\psi = 0, \beta d = \pi$ , principal maximum is at  $\phi = \pm \frac{\pi}{2}$ . If  $N = 5$ , nulls are at  $\phi = \pm \cos^{-1} \left( \frac{2n}{5} \right)$ , or  $\phi = \pm 66.4^\circ, \pm 36.9^\circ, \pm 113.6^\circ, \pm 143.1^\circ$ . The radiation pattern is seen to form *lobes* of the antenna radiation pattern. The largest lobe is called the main lobe, while the smaller lobes are called side lobes. Since  $\psi = 0$ , the radiated fields in the  $y$  direction are in phase and the peak of the radiation lobe is in the  $y$  direction or in the broadside direction (see Figure 27.3 for the definition of broadside and endfire.). Hence, this is called a *broadside array*. The radiation pattern of such an array is shown in Figure 27.3.

**Case II.**  $\psi = \pi, \beta d = \pi$ , principal maximum is at  $\phi = 0, \pi$ . If  $N = 4$ , nulls are at  $\phi = \pm \cos^{-1} \left( \frac{n}{2} - 1 \right)$ , or  $\phi = \pm 120^\circ, \pm 90^\circ, \pm 60^\circ$ . Since the sources are out of phase by  $180^\circ$ , and  $N = 4$  is even, the radiation fields cancel each other in the broadside, but add in the  $x$  direction or the end-fire direction. This is called the *endfire array*. Figure 27.4 shows the radiation pattern of such an array.

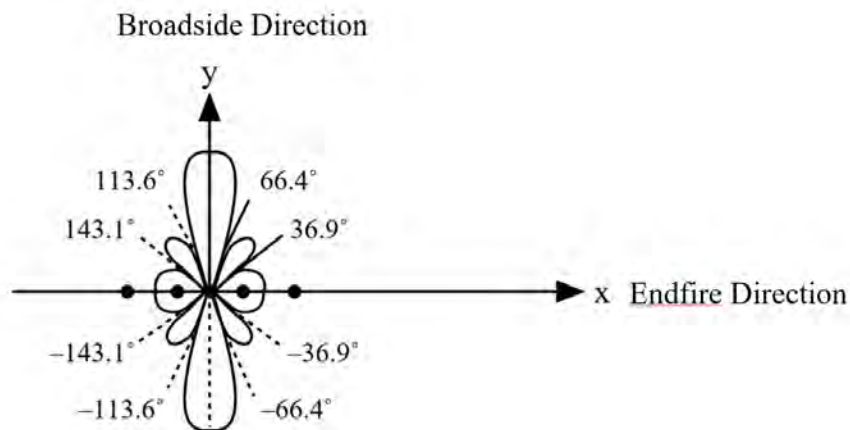


Figure 27.3: The radiation pattern of a five-element broadside array. The broadside and endfire directions of the array are also labeled.

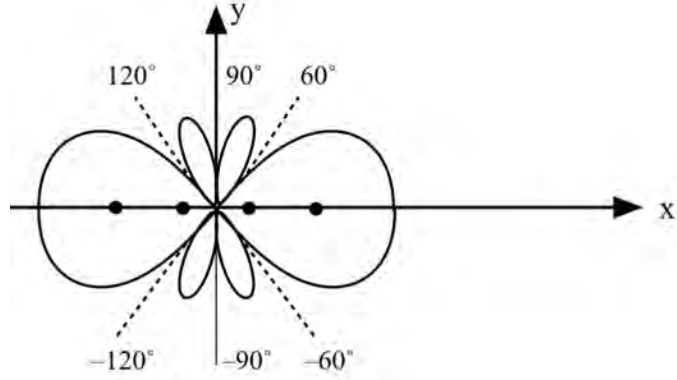


Figure 27.4: By changing the phase of the linear array, the radiation pattern of the antenna array can be changed to become an endfire array.

From the above examples, it is seen that the interference effects between the different antenna elements of a linear array focus the power in a given direction. We can use linear array to increase the directivity of antennas. Moreover, it is shown that the radiation patterns can be changed by adjusting the spacings of the elements as well as the relative phase shift between them. The idea of antenna array design is to make the main lobe of the pattern to be much higher than the side lobes so that the radiated power of the antenna is directed along the main lobe rather than the side lobes. So side-lobe level suppression is an important goal of designing a highly directive antenna. Also, by changing the phase of the antenna elements in real time, the beam of the antenna can be steered in real time with no moving parts.

## 27.2 Validity of the Far-Field Approximation

In making the far-field approximation in (27.1.3), it will be interesting to ponder when the far-field approximation is valid? That is, when we can approximate

$$e^{-j\beta|\mathbf{r}-\mathbf{r}'|} \approx e^{-j\beta r + j\beta \mathbf{r}' \cdot \hat{\mathbf{r}}} \quad (27.2.1)$$

to arrive at (27.1.3). This is especially important because when we integrate over  $\mathbf{r}'$ , it can range over large values especially for a large array. In this case,  $\mathbf{r}'$  can be as large as  $(N-1)d$ . The above approximation is important also because it tells when the field generated by an array antenna becomes a spherical wave.

To answer this question, we need to study the approximation in (27.2.1) more carefully. First, we have

$$|\mathbf{r}-\mathbf{r}'|^2 = (\mathbf{r}-\mathbf{r}') \cdot (\mathbf{r}-\mathbf{r}') = r^2 - 2\mathbf{r} \cdot \mathbf{r}' + r'^2 \quad (27.2.2)$$

We can take the square root of the above to get

$$|\mathbf{r} - \mathbf{r}'| = r \left( 1 - \frac{2\mathbf{r} \cdot \mathbf{r}'}{r^2} + \frac{r'^2}{r^2} \right)^{1/2} \quad (27.2.3)$$

Next, we use the Taylor series expansion to get, for small  $x$ , that

$$(1 + x)^n \approx 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots \quad (27.2.4)$$

or that

$$(1 + x)^{1/2} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots \quad (27.2.5)$$

We can apply this approximation by letting

$$x \cong -\frac{2\mathbf{r} \cdot \mathbf{r}'}{r^2} + \frac{r'^2}{r^2}$$

To this end, we arrive at<sup>3</sup>

$$|\mathbf{r} - \mathbf{r}'| \approx r \left[ 1 - \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} + \frac{1}{2} \frac{r'^2}{r^2} - \frac{1}{2} \left( \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} \right)^2 + \dots \right] \quad (27.2.6)$$

In the above, we have not kept every term of the  $x^2$  terms by assuming that  $r'^2 \ll \mathbf{r}' \cdot \mathbf{r}$ , and terms much smaller than the last term in (27.2.6) can be neglected.

We can multiply out the right-hand side of the above to further arrive at

$$\begin{aligned} |\mathbf{r} - \mathbf{r}'| &\approx r - \frac{\mathbf{r} \cdot \mathbf{r}'}{r} + \frac{1}{2} \frac{r'^2}{r} - \frac{1}{2} \frac{(\mathbf{r} \cdot \mathbf{r}')^2}{r^3} + \dots \\ &= r - \hat{\mathbf{r}} \cdot \mathbf{r}' + \frac{1}{2} \frac{r'^2}{r} - \frac{1}{2r} (\hat{\mathbf{r}} \cdot \mathbf{r}')^2 + \dots \end{aligned} \quad (27.2.7)$$

The last two terms in the last line of (27.2.7) are of the same order.<sup>4</sup> Moreover, their sum is bounded by  $r'^2/(2r)$  since  $\hat{\mathbf{r}} \cdot \mathbf{r}'$  is always less than  $r'$ . Hence, the far field approximation is valid if

$$\beta \frac{r'^2}{2r} \ll 1 \quad (27.2.8)$$

In the above,  $\beta$  is involved because the approximation has to be valid in the exponent, namely  $\exp(-j\beta|\mathbf{r} - \mathbf{r}'|)$  where  $\beta$  multiplies  $|\mathbf{r} - \mathbf{r}'|$  or its approximation. If (27.2.8) is valid, then

$$e^{j\beta \frac{r'^2}{2r}} \approx 1$$

and thus, the first two terms on the right-hand side of (27.2.7) suffice to approximate  $|\mathbf{r} - \mathbf{r}'|$  on the left-hand side, which are the two terms we have kept in the far-field approximation.

<sup>3</sup>The art of making such approximation is called perturbation expansion [46].

<sup>4</sup>The math parlance for saying that these two terms are approximately of the same magnitude as each other.

### 27.2.1 Rayleigh Distance

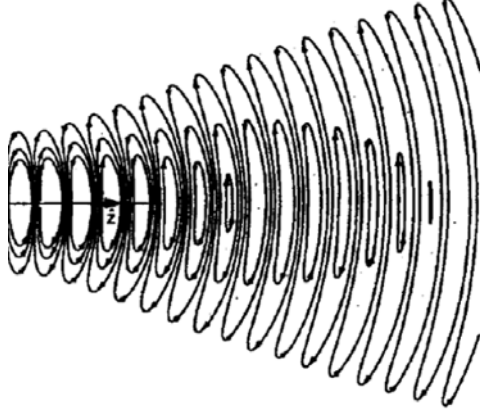


Figure 27.5: The right half of a Gaussian beam displays the physics of the near field, the Fresnel zone, and the far zone. In the far zone, the field behaves like a spherical wave (courtesy of [84]).

If we have an infinite time-harmonic current sheet, it can be shown that by matching boundary conditions, it will launch plane waves on both sides of the current sheet [33][p. 652]. Thus if we have an aperture antenna like the opening of a waveguide that is much larger than the wavelength, it will launch a wave that is almost like a plane wave from it. Thus, when a wave field leaves an aperture antenna, it can be approximately described by a Gaussian beam [84] (see Figure 27.5). Near to the antenna aperture, or in the near zone, it is approximately a plane wave with wave fronts parallel to the aperture surface. Far from the antenna aperture, or in the far zone, the field behaves like a spherical wave, with its typical wave front. In between, we are in the Fresnel zone.

Consequently, after using that  $\beta = 2\pi/\lambda$ , for the far-field approximation to be valid, we need (27.2.8) to be valid, or that

$$r \gg \frac{\pi}{\lambda} r'^2 \quad (27.2.9)$$

If the aperture of the antenna is of radius  $W$ , then  $r' < r_{\max} \cong W$  and the far field approximation is valid if

$$r \gg \frac{\pi}{\lambda} W^2 = r_R \quad (27.2.10)$$

If  $r$  is larger than this distance, then an antenna beam behaves like a spherical wave and starts to diverge. This distance  $r_R$  is also known as the Rayleigh distance. After this distance, the wave from a finite size source resembles a spherical wave which is diverging in all directions (see Figure 27.5). Also, notice that the shorter the wavelength  $\lambda$ , the larger is this distance before the far field approximation can be made.



This also explains why a laser pointer works. A laser pointer light can be thought of radiation from a finite size source located at the aperture of the laser pointer as shall be shown using equivalence theorem later. The laser pointer beam remains collimated for quite a distance, before it becomes a divergent beam or a beam with a spherical wave front.

In some textbooks [33], it is common to define acceptable phase error to be  $\pi/8$ . The Rayleigh distance is the distance beyond which the phase error is below this value. When the phase error of  $\pi/8$  is put on the right-hand side of (27.2.8), one gets

$$\beta \frac{r'^2}{2r} \approx \frac{\pi}{8} \quad (27.2.11)$$

Using the approximation, the Rayleigh distance is defined to be

$$r_R = \frac{2D^2}{\lambda} \quad (27.2.12)$$

where  $D = 2W$  is the diameter of the antenna aperture. This concept is important in both optics and microwave.

### 27.2.2 Near Zone, Fresnel Zone, and Far Zone

Therefore, when a source radiates, the radiation field is divided into the near zone, the Fresnel zone, and the far zone (also known as the radiation zone, or the Fraunhofer zone in optics). The Rayleigh distance is the demarcation boundary between the Fresnel zone and the far zone. The larger the aperture of an antenna array is, the further one has to be in order to reach the far zone of an antenna. This distance becomes larger too when the wavelength is short. In the far zone, the far field behaves like a spherical wave, and its radiation pattern is proportional to the Fourier transform of the current.

In some sources, like the Hertzian dipole, in the near zone, much reactive energy is stored in the electric field or the magnetic field near to the source. This near zone receives reactive power from the source, which corresponds to instantaneous power that flows from the source, but is return to the source after one time harmonic cycle. Thus, reactive power corresponds to energy that sloshes back and forth between the source and the near field. Hence, a Hertzian dipole has input impedance that looks like that of a capacitor, because much of the near field of this dipole is the electric field.

The field in the far zone carries power that radiates to infinity. As a result, the field in the near zone decays rapidly, because it stores energy and carries reactive power that needs to be exchanged with the source. But the field in the far zone decays as  $1/r$  for energy conservation. Moreover, the far field convects energy to infinity.

